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## Quantum Mechanics for Thermal Equilibriums

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Thermal equilibrium states which satisfy the principle of equal *a priori* probability are investigated in the framework of quantum mechanics.

### 1. Introduction

There is a crucial gap between quantum mechanics and experimental results, which is known as the problem for the quantum theory of measurements. That is to say, quantum mechanical states are described by pure states, whereas experimental ones are done by mixed states. In order to interpret the experimental results rigorously, we have to derive the disappearance of the interference terms in quantum mechanical states. The same problem exists between quantum mechanical states and thermal equilibrium states. For instance, the former is described by pure states as

$$\rho_\psi = \sum_n |c_n|^2 |n\rangle\langle n| + \sum_{n \neq m} c_n c_m^* |n\rangle\langle m| \quad (1)$$

for the state  $|\psi\rangle = \sum_n c_n |n\rangle$  where the complete set  $|n\rangle$  fulfills the equations  $H_0 |n\rangle = E_n |n\rangle$  and  $\langle n|m\rangle = \delta_{n,m}$ , whereas the latter is done by mixed states such that

$$\rho_{stat} = \sum_n e^{-\beta E_n} |n\rangle\langle n| / Z \quad (2)$$

for canonical ensembles in statistical mechanics, where  $\beta = (kT)^{-1}$  ( $k$  = Boltzmann's constant) and  $Z = \text{Tr}(\sum_n e^{-\beta E_n} |n\rangle\langle n|)$  (the partition function of the canonical ensemble).

The main difference between them is represented by the following two problems:

- (1) In order to drive the disappearance of the interference terms some kinds of decoherence mechanisms must exist in the thermal equilibrium states. (Decoherence problem)
- (2) The relations (canonical-ensemble relations)  $|c_n|^2 \propto e^{-\beta E_n}$ , for  $\forall n$  must be fulfilled

in  $\rho_{stat}$  for the canonical ensembles. (Canonical-ensemble problem)

The first problem is quite similar to that for describing the experimental results in the quantum theory of measurements, while the second one is a purely thermodynamical problem. The apparent difference between the two states is represented by the existence of heat baths in thermal equilibriums. How to write down the heat baths in the framework of quantum mechanics has not yet clearly been known in general cases. The quantum description of the heat baths will be the first step to describe thermal equilibriums in quantum mechanics. In statistical mechanics total systems including the heat baths and the objective matters (the objects) are always written down in terms of microcanonical ensembles which describe thermally isolated systems. We shall start from the quantum description of such thermally isolated systems. The basic assumption for the thermally isolated systems is the principle of equal *a priori* probability.

For the states given in (1) the principle are expressed by the relations  $|c_n|^2 = |c_m|^2$ , for  $E_n = E_m$ . It is obvious that general quantum mechanical states do not fulfill the relations and also the states fulfilling the relations generally span only a special subspace of the whole physical space  $\mathcal{H}$  spanned by the complete set of  $H_0$ . (Hereafter we call the subspace the thermal subspace of  $H_0$  and write it as  $\mathcal{H}_{thermal}$ .) The projection of  $\mathcal{H}$  to  $\mathcal{H}_{thermal}$  will be one of the important problems in the quantum-mechanical description of the thermal equilibriums as same as the decoherence problem. It is noted that the principle is fulfilled by the states having energies within the certain experimental uncertainty  $\delta E$ , that is, the relations  $|c_n|^2 = |c_m|^2$  must be fulfilled among the states having the energies  $E_m$  within  $E_n - \delta E < E_m < E_n + \delta E$  ( $E_n \gg \delta E$ ). This means that interaction  $H_I$ , of which contributions to the energy of the system are small enough to be ignored in comparison with those of the original Hamiltonian  $H_0$  such that  $\langle m|H_I|m \rangle < \delta E$ , may be included in the total Hamiltonian  $H = H_0 + H_I$ . It will be possible that the physical space of the total Hamiltonian  $H$  coincides with the thermal subspace of  $H_0$ ,  $\mathcal{H}_{thermal}$ . In this description the interaction  $H_I$  will be taken to be negligible in

thermodynamical limits. That is to say, in the limit all the contributions arising from  $H_I$  are involved in the energy uncertainty  $\delta E$  and then the thermal subspace of  $H_0$  seems to be realized. When we write the limit by  $H_I \rightarrow 0$ , we should not forget about the fact that some physical quantities induced from the introduction of  $H_I$  become un-measurable in the limit. Thus we have to introduce some kinds of average operations with respect to such un-measurable quantities. It is important that such average will be represented by the partial trace operation (the internal trace  $\text{Tr}_I$ ) with respect to such un-measurable quantities, which plays an essential role to realize the decoherence in the quantum theory of measurements. Now we can expect that the introduction of the thermal interaction  $H_I$  has the possibility for solving the two problems, the decoherence problem and the canonical-ensemble problems, simultaneously in the limit  $H_I \rightarrow 0$ . Actually the author presented a simple model to realize the above scenario for a  $N$ -harmonic oscillator ( $N$ -particle) system described by the original Hamiltonian  $H_0 = \epsilon \sum_{i=1}^N a_i^\dagger a_i$ . In this paper I shall discuss the general scheme for the model realizing thermal equilibriums not only for bosonic oscillators but for fermionic ones as well.

## 2. Quantum description of thermal equilibriums for $N$ -oscillator system with one energy scale

Here we shall study quantum mechanics for thermal equilibriums for a simple oscillator system with one energy scale. [1-4]

2-1. Model: The model is described by the system composed of  $N$  oscillators (or  $N$  particle) which are described by the Hamiltonian  $H_0$  as

$$H_0 = \epsilon \sum_{j=1}^N a_j^\dagger a_j, \quad (3)$$

where  $a_j^\dagger$  and  $a_j$  are, respectively, the creation and annihilation operators and follow the commutation relations  $[a_i, a_j^\dagger] = \delta_{ij}$ . The phases  $\theta_j$  are introduced to the oscillators by using the operators  $a_j(\theta) = a_j e^{-i\theta_j}$  and  $a_j^\dagger(\theta) = a_j^\dagger e^{i\theta_j}$  in stead of  $a_j^\dagger$  and  $a_j$ . The

Hamiltonian  $H_0$  and the commutation relations do not change in such a replacement.

The eigenstates of  $H_0$  with the eigenvalues  $E_M = \epsilon M$  ( $M = 0, 1, 2, \dots$ ) are given as

$$|M; [n_i], [\theta_j]\rangle \equiv |M; n_1, n_2, \dots, n_N; \theta_1, \theta_2, \dots, \theta_N\rangle = \prod_{i=1}^N |n_i, \theta_i\rangle \delta_{\sum_{i=1}^N n_i, M}, \quad (4)$$

where  $|n_i, \theta_i\rangle = |n_i\rangle e^{in_i\theta_i}$  fulfills the equations  $a_j^\dagger(\theta)|n_j, \theta_j\rangle = \sqrt{n_j+1}|n_j+1, \theta_j\rangle$ ,  $a_j(\theta)|n_j, \theta_j\rangle = \sqrt{n_j}|n_j-1, \theta_j\rangle$  and  $a_j^\dagger(\theta)a_j(\theta)|n_j, \theta_j\rangle = n_j|n_j, \theta_j\rangle$ . The states satisfying the principle of equal *a priori* probability for the eigenstates of  $H_0$  are given as

$$|M, N, [\theta]\rangle = \sum_{p[n_i]} |M; [n_i], [\theta_j]\rangle / \sqrt{W(M, N)}, \quad (5)$$

where the sum should be taken over all the different combinations of  $[n_i] = (n_1, n_2, \dots, n_N)$  and the number of the combinations  $W(M, N)$  is given by  $W(M, N) = \frac{(M+N-1)!}{M!(N-1)!}$ .

2-2. Relative-phases interaction: We introduce the new operators;

$$\alpha_j^\dagger(\theta) \equiv a_j^\dagger(\theta)(\sqrt{\hat{N}_j+1})^{-1}, \quad \alpha_j(\theta) \equiv (\sqrt{\hat{N}_j+1})^{-1}a_j(\theta) \quad \text{for } i = 1, \dots, N, \quad (6)$$

where  $\hat{N}_j = a_j^\dagger a_j$  is the excitation-number operator for the  $j$ th oscillator. We see that  $\alpha_j^\dagger(\theta)|n_j, \theta_j\rangle = |n_j+1, \theta_j\rangle$ , for  $\forall n_j$  and  $\alpha_j(\theta)|n_j, \theta_j\rangle = |n_j-1, \theta_j\rangle$ , for  $\forall n_j \geq 1$  except  $\alpha_j(\theta)|0\rangle = 0$ . The relative-phase interaction with the energy scale  $\epsilon_g$  is written as

$$H_I^{(1)r}([\theta]) = \epsilon_g \frac{\hat{N}}{N} \sum_j \left( \frac{1}{N} \sum_k \alpha_j^\dagger \alpha_k e^{i\theta_{jk}} + |0\rangle_j \langle 0| \right), \quad (7)$$

where  $\alpha_j^\dagger = \alpha_j^\dagger(0)$ ,  $\alpha_k = \alpha_k(0)$ ,  $\hat{N} = \sum_{j=1}^N \hat{N}_j$  (the total excitation-number operator) and the relative phases  $\theta_{jk} = \theta_j - \theta_k$ . We obtain the relation

$$H|M, N, [\theta]\rangle = (\epsilon + \epsilon_g)M|M, N, [\theta]\rangle \quad (8)$$

for  $H = H_0 + H_I^{(1)r}([\theta])$ . We see that the physical space of the total Hamiltonian  $H$  is spanned by the set of the states  $|M, N, [\theta]\rangle$  for  $M = 0, 1, 2, \dots$ , which is nothing but the thermal subspace of  $H_0$ . Thermal limit will be described by the limit  $\epsilon_g \rightarrow 0$ , where the eigenvalues of  $H$  can approximate to those of  $H_0$ .

2-3. Decoherence: In order to derive microcanonical ensembles the decoherence with respect to the eigenstates of  $H_0$  must be derived. It can be realized by the average with respect to the phases, which are un-measurable in the thermal limit  $\epsilon_g \rightarrow 0$ , such that

$$\begin{aligned}\rho_{eff} &= \prod_{j=1}^N \int_0^{2\pi} \frac{d\theta_j}{2\pi} |M, N, [\theta] \rangle \langle M, N, [\theta]| \\ &= \frac{1}{W(M, N)} \sum_{p[n_i]} |M : [n_i] \rangle \langle M; [n_i]|.\end{aligned}\quad (9)$$

2-4. A useful expression on the thermal subspace  $\mathcal{H}_{thermal}$ : Let us introduce a pair of useful operators  $(\beta, \bar{\beta})$ , which are defined by  $\beta \equiv \sum_{j=1}^N \alpha_j(\theta)/N$  and  $\bar{\beta} \equiv \sum_{j=1}^N \bar{\alpha}_j(\theta)$  with  $\bar{\alpha}_j(\theta) \equiv a_j^\dagger e^{i\theta_j} \sqrt{\hat{N}_j + 1}$ . We easily see that on the thermal subspace  $\mathcal{H}_{thermal}$   $\beta|M, N, [\theta] \rangle = \sqrt{\frac{M}{M+N-1}}|M-1, N, [\theta] \rangle$ ,  $\bar{\beta}|M, N, [\theta] \rangle = \sqrt{(M+1)(M+N)}|M+1, N, [\theta] \rangle$  and also  $[\beta, \bar{\beta}] = 1$  and  $\bar{\beta}\beta|M, N, [\theta] \rangle = M|M, N, [\theta] \rangle$ . We shall see that they are very useful in the following discussions. For example we can rewrite  $H_I^{(1)r}([\theta])$  in the following very simple form on  $\mathcal{H}_{thermal}$ :

$$H_I^{(1)r}([\theta]) = \epsilon_g \bar{\beta} \beta.$$

2-5. Derivation of canonical ensembles and temperature: We divide the total system into two groups,  $b_j \equiv a_j$  ( $j = 1, 2, \dots, N_b$ ) ( $b$ -group) and  $d_{j-N_b} \equiv a_j$  for  $j \geq N_b + 1$  ( $d$ -group). The eigenstates of the total Hamiltonian  $|M, N, [\theta] \rangle$  can be written in terms of the product of the eigenstates of the groups as

$$|M, N, [\theta][\phi] \rangle = \sum_{M_b=0}^M \sum_{M_d=0}^M \delta_{M_b+M_d, M} \frac{\sqrt{W(M_b, N_b)}}{\sqrt{W(M, N)}} |M_b, N_b, [\theta] \rangle_b \otimes \sqrt{W(M_d, N_d)} |M_d, N_d, [\phi] \rangle_d, \quad (10)$$

where  $|M_b, N_b, [\theta] \rangle_b$  and  $|M_d, N_d, [\phi] \rangle_d$  are, respectively, the eigenstate on the thermal subspace of the  $b$ -group and that of the  $d$ -group. Hereafter we shall treat the  $b$ -group as the heat bath and the  $d$ -group as the object. Then the  $b$ -group is taken to be much larger than the  $d$ -group, i.e.  $N_b \gg N_d$  and  $M_b \gg M_d$ .

The effective density matrix for the object can be derived by performing the internal trace( $\text{Tr}_I$ ) for the heat-bath variables such that

$$\rho_{eff} = \sum_{M_b=0}^M \sum_{M_d=0}^M \delta_{M_b+M_d,M} \frac{W(M_b, N_b)}{W(M, N)} \sum_{p[m_j]} |M_1; [m_j] \rangle_d \langle M_1; [m_j]|. \quad (11)$$

In the above equation the integration over the phases  $[\phi]$  is also taken, because the interaction of the object with the heat bath is small enough to ignore, i.e.  $\epsilon_g \ll \epsilon$ .

The canonical ensemble is realized at the maximum of the probability  $P_M(M_b) = \frac{W(M_b, N_b)W(M-M_b, N-N_b)}{W(M, N)}$  for finding the state  $|M_b, N_b, [\theta] \rangle_b \otimes |M_d, N_d, [\phi] \rangle_d$  in  $|M, N, [\theta][\phi] \rangle$ . At the maximum we have the relation

$$\frac{1}{W(M_b, N_b)} \frac{\partial W(M_b, N_b)}{\partial M_b} = \frac{1}{W(M_d, N - N_b)} \frac{\partial W(M_d, N - N_b)}{\partial M_d} \Big|_{M_d=M-M_b}. \quad (12)$$

Thus the temperature is introduced as the common physical quantity between the two systems. We can estimate the change of the coefficients with respect to the change of the energy of the object, that is,  $\Delta E^{ob} = \epsilon \Delta M_d$  with  $\Delta M_d \sim M_d \ll M, M_b$ , as

$$\frac{W(M_b - \Delta M_d, N_b)}{W(M_b, N_b)} \sim \left( \frac{M_b}{M_b + N_b} \right)^{\Delta M_d} = e^{-\frac{1}{\epsilon} \Delta E^{ob} \ln(1+N_b/M_b)}. \quad (13)$$

We can determine the temperature by using the quantities of the heat bath as

$$kT = \epsilon (\ln(1 + N_b/M_b))^{-1}. \quad (14)$$

We obtain the density matrix for canonical ensembles in the limit  $M \sim M_b, N \sim N_b \rightarrow \infty$

$$\rho_{eff} \simeq \sum_{M_d=0}^{\infty} e^{-\beta E_{M_d}^{ob}} \sum_{p[m_i]} |M_d; [m_i] \rangle_d \langle M_d; [m_i]| / Z_{N_d}. \quad (15)$$

### 3. Thermal equilibriums for systems with different energy scales

At first let us study the thermal equilibrium between two subsystems carrying different energy scales. The free Hamiltonian  $H_0$  and the interaction  $H_I^{(1)r}([\theta])$  are written as

$$H_0 = \epsilon_1 \sum_{j_1=1}^{N_1} a_{j_1}^\dagger a_{j_1} + \epsilon_2 \sum_{j_2=1}^{N_2} a_{j_2}^\dagger a_{j_2}, \quad H_I^{(1)r}([\theta]) = H_I^{(1)r}([\theta]; 1) + H_I^{(2)r}([\theta]; 2), \quad (16)$$

where  $\epsilon_1 \neq \epsilon_2$  and  $H_I^{(1)r}([\theta]; k)$  is the interaction for the  $k$ th subsystem ( $k = 1, 2$ ) given in the section 2. Since the total system is treated as an isolated system and then the total energy must be conserved, there must be a relation between the energy scales such that

$$l_1(\epsilon_1 + \epsilon_{1g}) = l_2(\epsilon_2 + \epsilon_{2g}), \quad \text{for } l_1, l_2 = \text{positive integers.} \quad (17)$$

3-1. Interaction between two subsystems with the relation  $\epsilon_2 = 2\epsilon_1$ : As the simplest example we start from the case for  $\epsilon_1 = \epsilon_2/2$  and we shall use the notations  $\epsilon_{1g} \equiv g_1\epsilon_1$  and  $\epsilon_{2g} \equiv g_1\epsilon_2$  so as to satisfy the above relation. The thermal interaction will include two different types of the interactions, that is, one is the thermal interaction  $H_I^{(1)r}([\theta])$  interpreting the equilibrium for each subsystem with one energy scale and the other the interaction  $H_I^{(2)}$  deriving the equilibrium between the two subsystems with the different energy scales. Taking account of the effects of  $H_I^{(1)r}$ , we may start from the states which are on the direct product of the thermal subspaces,  $\mathcal{H}_{thermal}^1 \otimes \mathcal{H}_{thermal}^2$  and then the states may generally be written by the superposition of the eigenstates of  $H_0 + H_I^{(1)r}([\theta])$  such that  $|M; N_1, [\theta^1]; N_2, [\theta^2] \rangle = \sum_{m=0}^M C_m |2m, N_1, [\theta^1] \rangle \otimes |M-m, N_2, [\theta^2] \rangle$ , where  $|2m, N_1, [\theta^1] \rangle$  and  $|M-m, N_2, [\theta^2] \rangle$  are, respectively, the thermal equilibrium states for the two subsystems characterized by  $\epsilon_1$  and  $\epsilon_2$ . Hereafter we shall discuss on the direct product space and use the operators defined there. In general the states do not satisfy the principle of equal *a priori* probability. Only the states on the subspace  $\mathcal{H}_{thermal}^{1,2}$  of  $\mathcal{H}_{thermal}^1 \otimes \mathcal{H}_{thermal}^2$ , which are given as  $|M; N_1, [\theta^1]; N_2, [\theta^2] \rangle = \sum_{m=0}^M \frac{\sqrt{W(2m, N_1)} |2m, N_1, [\theta^1] \rangle \otimes \sqrt{W(M-m, N_2)} |M-m, N_2, [\theta^2] \rangle}{\sqrt{W_2(M, N_1, N_2)}}$ , satisfy the principle. How to derive the subspace  $\mathcal{H}_{thermal}^{1,2}$  by introducing the new interaction  $H_I^{(2)}$  is the present problem.  $H_I^{(2)}$  must preserve the thermal equilibriums in the two subsystems. This means that  $H_I^{(2)}$  should be described by the operators defined on  $\mathcal{H}_{thermal}^1 \otimes \mathcal{H}_{thermal}^2$ . The most convenient way is to use the operators  $\beta$  and  $\bar{\beta}$  defined on  $\mathcal{H}_{thermal}$ .

Following the same procedure developed in the section 2, we introduce the new phase  $\varphi_1$  to the eigenstates of  $\mathcal{H}_{thermal}^1$  and  $\varphi_2$  to those of  $\mathcal{H}_{thermal}^2$ . In place of  $\beta_j$  and  $\bar{\beta}_j$  on



$\mathcal{H}_{thermal}^j$  we use the operators containing the new phase  $\varphi_j$

$$\beta_j(\varphi) = \beta_j e^{-i\varphi_j}, \quad \bar{\beta}_j(\varphi) = \bar{\beta}_j e^{i\varphi_j} \quad (j = 1, 2). \quad (18)$$

We see that this replacement induces the total phase  $M\varphi_j$  [1,5] to the eigenstate of the  $j$ th subsystem with  $\epsilon_j$ . Here we replace  $\bar{\beta}_j(\varphi)$  by  $\hat{\beta}_j(\varphi)$  which is defined by

$$\hat{\beta}_j(\varphi) \equiv \frac{1}{\hat{M}_j} \bar{\beta}_j(\varphi) \quad \text{for } j = 1, 2, \quad (19)$$

where  $\hat{M}_j = \bar{\beta}_j(\varphi)\beta_j(\varphi)$  with  $\hat{M}_j|M, N_j, [\theta]_j, \varphi_j \rangle = M|M, N_j, [\theta]_j, \varphi_j \rangle$  for the states,  $|M, N_j, [\theta]_j, \varphi_j \rangle = |M, N_j, [\theta]_j \rangle e^{iM\varphi_j}$ , having the total phase  $M\varphi_j$ . We obtain

$$\beta_j(\varphi)|M \rangle_j = \sqrt{\frac{W(M-1, N_j)}{W(M, N_j)}}|M-1 \rangle_j, \quad \hat{\beta}_j(\varphi)|M \rangle_j = \sqrt{\frac{W(M+1, N_j)}{W(M, N_j)}}|M+1 \rangle_j \quad (20)$$

and then we have  $\hat{\beta}_j(\varphi)\beta_j(\varphi)|M \rangle_j = |M \rangle_j$ , where the abbreviations,  $|M \rangle_j \equiv |M, N_j, [\theta]_j, \varphi_j \rangle$ , for the eigenstates of  $H_0^j + H_I^{(j)r}([\theta]; j)$  are used. It must be stressed that finding the operators  $\beta_j(\varphi)$  and  $\hat{\beta}_j(\varphi)$  satisfying the above relations is essential to construct thermal interactions in the following discussions. We introduce the operators

$$B_\varphi \equiv \frac{1}{2}[(\beta_1(\varphi))^2 + \beta_2(\varphi)], \quad \bar{B}_\varphi \equiv [\frac{\hat{M}_1}{2}(\hat{\beta}_1(\varphi))^2 + \hat{M}_2\hat{\beta}_2(\varphi)]. \quad (21)$$

The thermal interaction can be written in terms of the relative phase  $\bar{\varphi} = 2\varphi_1 - \varphi_2$  as

$$H_I^{(2)}(\varphi) = g_2 \epsilon (2\bar{B}_\varphi B_\varphi), \quad (22)$$

where  $\epsilon \equiv \epsilon_1$ . We easily obtain  $H_I^{(2)}(\varphi)|M; 2M, [\varphi] \rangle = 2g_2 M \epsilon |M; 2M, [\varphi] \rangle$  for the states on  $\mathcal{H}_{thermal}^{1,2}$  with the energy  $E = 2M\epsilon = M\epsilon_2$ , that is,

$$|M; 2M, [\varphi] \rangle = \sum_{m=0}^M \frac{\sqrt{W(2m, N_1)}|2m \rangle_1 \otimes \sqrt{W(M-m, N_2)}|M-m \rangle_2}{\sqrt{W_2(M; 2)}}, \quad (23)$$

where  $W_2(M; 2) = \sum_{m=0}^M W(2m, N_1)W(M-m, N_2)$ . In the general case including the states with  $E = (2M+1)\epsilon$ ,  $|M; 2M+1, [\varphi] \rangle = \sum_{m=0}^M \frac{\sqrt{W(2m+1, N_1)}|2m+1 \rangle_1 \otimes \sqrt{W(M-m, N_2)}|M-m \rangle_2}{\sqrt{W_2'(M; 2)}}$ ,

we have to reformulate  $H_I^{(2)}$  as

$$H_I^{(2)r}(\varphi) = H_I^{(2)}(\varphi) + g_2 \epsilon \sum_{l=0}^1 l |l >_1 \quad 1 < l|. \quad (24)$$

The correction is due to the properties  $(\beta_1(\varphi))^2 |0 >_1 = (\beta_1(\varphi))^2 |1 >_1 = 0$ . We have the relation  $H|M; M', [\varphi] > = (1 + g_1 + g_2) \epsilon M' |M; M', [\varphi] >$  for the total Hamiltonian  $H = H_0 + H_I^{(1)r}([\theta]) + H_I^{(2)r}(\varphi)$ , where  $M' = 2M$  and  $2M + 1$ . Thus we see that the thermal subspace  $\mathcal{H}_{thermal}^{1,2}$  is the physical space of the total Hamiltonian  $H$ .

3-2. General cases: For the details of the general cases, see ref.4.

3-3. Temperature between the two subsystems: Following the same procedure carried out in 2-5, we derive  $T_1 = T_2$ . And then we get the relation between the two systems

$$(1 + N_1/M_1)^{l_1} = (1 + N_2/M_2)^{l_2}. \quad (25)$$

This relation is reasonable, because it implies that the mean excitation number  $(M_j/N_j)$  for the lower energy scale is larger than that for the higher one.

3- 4. Thermal equilibriums for the systems with more than three energy scale: We can explicitly write the thermal interactions for 3- and 4-point vertices. (In detail, see ref. 4.) And we have the relation among the different systems

$$(1 + N_1/M_1)^{l_1} = (1 + N_2/M_2)^{l_2} = (1 + N_3/M_3)^{l_3} = \dots \quad (26)$$

#### 4. Fermionic systems

4-1. Fermionic system with one energy scale  $\epsilon$ : It is easily carried out by introducing anti-commuting oscillators  $c_j$  with  $j = 1, 2, \dots, N$ , that is,  $[c_j(\theta), c_j^\dagger(\theta)]_+ = 1$   $[c_j(\theta), c_j(\theta)]_+ = 0$ , where  $[a, b]_+ \equiv ab + ba$ ,  $c_j(\theta) = c_j e^{-i\theta_j}$  and  $c_j^\dagger(\theta) = c_j^\dagger e^{i\theta_j}$ . For the simplicity the usual bosonic-commutation relations between the oscillators belonging to the different site will be imposed in the following discussions such that  $[c_j, c_k^\dagger] = 0$  and so on. The original

Hamiltonian can be given by  $H_0 = \epsilon \sum_{j=1}^N c_j^\dagger c_j$ , of which eigenstates with the phases for the eigenvalue  $\epsilon M$  ( $M = 0, 1, 2, \dots$ ) are described in terms of direct products of number states  $(|n_j; f\rangle) |M; [n_j]; [\theta]; f\rangle = \prod_{j=1}^N |n_j; f\rangle e^{in_j \theta_j} \delta_{\sum_j n_j, M}$ . Here, of course,  $n_j = 0$  or 1 is only permissible. By introducing the totally symmetric operators

$$C_\theta = \frac{1}{\sqrt{N - \hat{N}}} \sum_{j=1}^N c_j(\theta), \quad C_\theta^\dagger = \sum_{j=1}^N c_j^\dagger(\theta) \frac{1}{\sqrt{N - \hat{N}}}, \quad (27)$$

where  $\hat{N} = \sum_{j=1}^N c_j^\dagger c_j$ , we can write the thermal interaction for one energy scale  $\epsilon$  as

$$H_I^{(1)}([\theta]; f) = g_1 \epsilon C_\theta^\dagger C_\theta. \quad (28)$$

The eigenstates of  $H_I^{(1)}([\theta]; f)$  are given by  $|M, N; f\rangle = \frac{1}{\sqrt{W(M, N; f)}} \sum_{p[n_j]} |M; [n_j]; [\theta]; f\rangle$ , where the number of the different permutations of  $[n_j] = (n_1, n_2, \dots, n_N)$  is evaluated as follows;  $W(M, N; f) = \frac{N!}{M!(N-M)!}$ . We can easily show the following equation;

$$H_I^{(1)}([\theta]; f) |M, N; f\rangle = g_1 \epsilon M |M, N; f\rangle, \quad (29)$$

where the relations

$$C_\theta |M, N; f\rangle = \sqrt{M} |M-1, N; f\rangle, \quad C_\theta^\dagger |M, N; f\rangle = \sqrt{M+1} |M+1, N; f\rangle \quad (30)$$

are used. Thus we see that the thermal subspace  $\mathcal{H}_{thermal}$  of the original Hamiltonian  $H_0$  is the physical space of the total Hamiltonian  $H = H_0 + H_I^{(1)}([\theta]; f)$ . Note here that the operators  $C_\theta$  and  $C_\theta^\dagger$ , respectively, play the roles of the annihilation and creation operators on the thermal subspace  $\mathcal{H}_{thermal}$  and obey the commutation relation  $[C_\theta, C_\theta^\dagger] = 1$  on  $\mathcal{H}_{thermal}$ .

Let us here introduce the operators on  $\mathcal{H}_{thermal}$

$$\gamma(\theta) = \frac{1}{N - \hat{N}} \sum_{j=1}^N c_j(\theta), \quad \hat{\gamma}(\theta) = \sum_{j=1}^N c_j^\dagger(\theta) \frac{1}{\hat{N} + 1} \quad (31)$$

which will be shown to be very useful in the discussions of thermal equilibriums for systems with different energy scales. The operators play the same roles of  $\beta(\theta)$  and  $\hat{\beta}(\theta)$  for bosonic systems, respectively.

The derivation of the decoherence and the canonical ensemble can be carried out by following the same procedures used for the bosonic case. The difference appears in the following expression of the temperature for the fermionic system;

$$kT_f = \epsilon(\ln(N/M - 1))^{-1}. \quad (32)$$

4-2. Fermionic systems with more than two energy scales: Thermal interactions among the systems can be written down as same as those for the bosonic cases given in §3. As was noted, the operators  $\gamma$  and  $\hat{\gamma}$  are very useful in the process. And we get the relation

$$(N_1/M_1 - 1)^{l_1} = (N_2/M_2 - 1)^{l_2} = (N_3/M_3 - 1)^{l_3} = \dots \quad (33)$$

4-3. Thermal equilibriums for fermion-boson mixed systems: The most interesting case is the interaction described by 3-point vertex, which will be a base for making field theory. Following the procedures presented in §3, we can get the relation among the mean-excitation numbers of the three subsystems

$$(N_1/M_1 - 1)^{l_1} = (N_b/M_b + 1)^{l_b} = (N_2/M_2 - 1)^{l_2}. \quad (34)$$

## 5. Remarks

We have seen that thermal equilibriums can be described in the framework of quantum mechanics. The most important point of this model is the existence of the thermal interactions. We have to detect the effects of the interactions in precise measurements.

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